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Modulated Simple Waves: An Approach To Attenuated Finite Amplitude Waves

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Abstract

This paper describes some of the techniques which are currently being used to investigate finite amplitude waves in elastic and viscoelastic materials. In particular we show how the simple wave solutions, which describe finite amplitude plane progressing waves in elastic materials, may be modified to describe the effects of reflection from boundaries, deformations behind curved wave fronts, and the effects of locally small damping mechanisms.

1. Introduction

The simplest significant example of a finite amplitude wavelike disturbance whose properties are thoroughly understood occurs when an infinite half-space of elastic material is loaded at its plane boundary by a time varying normal traction. If the mechanical response of the material is isotropic and homogeneous with respect to its state before the traction is applied then, if all sources of power are negligible compared with the mechanical power generated by the stresses, up to the instant when a shock forms the deformation is produced by a simple wave, (see Taylor [1]). These waves are fully amplitude dispersed: the wavelets at which the strain is invariant propagate at an invariant speed which is determined by the strain level carried. Both the stress level and material velocity are also invariant at each wavelet and can be computed from the strain by relations which are characteristic of the material and are independent of the variation in traction which produced the wave. In this paper we will show how the simple wave solutions can be used as building blocks to construct theories which describe the global effects of locally small non-uniformities and attenuating mechanisms such as weak shocks, material viscosity, and radial spreading of the disturbance.

2. Small Amplitude, Finite Rate Theory.

To describe one dimensional longitudinal disturbances in an elastic material let

$$x = x(X,t) \tag{2.1}$$

be the position at time t, measured from the boundary X=0 of the material, of the material particle X which in some reference state, when the material was at constant density ρ_0 and constant hydrostatic pressure p_0, was at a distance X from

the driven end. Let $T(X,t)$ be the traction measured from this reference state. Then, the momentum equation for the material relates T to the material velocity

$$u(X,t) = \frac{\partial x}{\partial t} \tag{2.2}$$

by

$$\frac{\partial T}{\partial X} = \rho_o \frac{\partial u}{\partial t} . \tag{2.3}$$

For isentropic deformations of an elastic material whose response is homogeneous with respect to the reference state

$$T = T(e) \tag{2.4}$$

is a known function of the strain

$$e = \frac{\partial x}{\partial X} - 1. \tag{2.5}$$

For small strains, we assume that

$$T = E[e+Me^2+0(e^3)]. \tag{2.6}$$

If we take

$$\bar{u} = u/a_o, \quad \text{and} \quad \bar{t} = a_o t \tag{2.7}$$

as the velocity and time measures, where

$$a_o = \sqrt{\frac{E}{\rho_o}} \tag{2.8}$$

is the sound speed in the reference configuration, then, if the bars are dropped, (2.3) reads

$$a^2 \frac{\partial e}{\partial X} = \frac{\partial u}{\partial t} \tag{2.9}$$

where, by (2.6),

$$a^2 = 1+2Me+O(e^2). \tag{2.10}$$

Equations (2.9) and (2.10) are supplemented by the continuity equation

$$\frac{\partial u}{\partial X} = \frac{\partial e}{\partial t} \tag{2.11}$$

which is obtained by eliminating x(X,t) from (2.2) and (2.5).

In any simple wave moving into a uniform region where $e\equiv 0$, irrespective of the form of the input signal

$$e(0,t) = h(t), \tag{2.12}$$

the material velocity can be calculated from the strain by the relation

$$u = -\int_0^e a(s)ds, = -c(e) \text{ say}. \tag{2.13}$$

For small strains

$$c(e) = e[1+\tfrac{1}{2}Me+O(e^2)]. \tag{2.14}$$

The variation of e(X,t) depends, of course, on h(t). It is given implicitly by

$$e = h(\alpha) \tag{2.15}$$

where the arrival time of the characteristic wavelet α=constant, which left X=0 at t=α, is given by

$$t = \alpha+X/a(e). \tag{2.16}$$

For small elongations e, which satisfy the condition that

$$|Me| << 1, \tag{2.17}$$

the traction T at any (X,t) can be calculated from e to a good approximation by the linear law

$$T = Ee \tag{2.18}$$

while, according to (2.13) and (2.14), the material velocity is also given to a good approximation by

$$u = -e. \tag{2.19}$$

It also <u>seems</u> reasonable to suppose that when (2.17) holds the variation in e(X,t) is governed to a first approximation by (2.9) and (2.11) with $a \equiv 1$. This is not necessarily so. The approximation e_L to e predicted by this model is

$$e_L = h(\alpha_L), \tag{2.20}$$

where

$$\alpha_L = t-X. \tag{2.21}$$

The mean value theorem of differential calculus together with conditions (2.15), (2.16), (2.18) and (2.19) imply that at any (X,t)

$$\begin{aligned} \left|\frac{e_L}{e} - 1\right| &= \left|\frac{h(\alpha_L)-h(\alpha)}{e}\right| \\ &= \left|\frac{(\alpha_L-\alpha)h'(\theta)}{e}\right|, \text{ for some } \theta \text{ between } \alpha \text{ and } \alpha_L, \\ &= D(e)\,|h'(\theta)|\,X, \end{aligned} \tag{2.22}$$

where

$$D(e) = \left|\frac{a-1}{ae}\right| = |M|[1+0(e)]. \tag{2.23}$$

According to (2.22) and (2.23), if (2.17) holds, only in the near field where

$$X \ll |M \max. |h'||^{-1} \qquad (2.24)$$

do the predictions of the linear theory of elasticity agree with those of the nonlinear theory.† Equivalently, the linear theory may be used to describe conditions over the range $0 \leq X \leq L$ only if (2.17) holds and if

$$\text{amplitude of imposed strain rate} \ll \frac{a_0}{|M|L}, = \omega_L \text{ say.} \qquad (2.25)$$

Theories for which the restriction (2.17) holds but for which the restriction (2.24) does not are called small amplitude, finite rate theories. Most of the general theories describing nonlinear effects in wavelike deformations are valid only in this limit. In what follows we restrict attention to such theories.

According to (2.14)-(2.16), to a first approximation conditions in a small amplitude, finite rate simple wave are described parametrically by

$$e = -u = h(\alpha), \qquad (2.26)$$

where the arrival time $t=t(\alpha,X)$ of the characteristic wavelet α is given by

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†Here we are discussing the role of the linear theory of elasticity as an approximation to the nonlinear theory of elasticity. In section 6 we show that if damping mechanisms are taken into account the range of validity of the linear theory of elasticity may be much wider than that indicated by the inequality (2.24).

$$t-X = \alpha-Mh(\alpha)X, = \alpha_L. \qquad (2.27)$$

Because the wave described by (2.26) and (2.27) is amplitude dispersed its profile distorts as it propagates. A measure of the distortion is the <u>incremental arrival time</u>

$$p(\alpha,X) = \frac{\partial t}{\partial \alpha} = \frac{\partial \alpha_L}{\partial \alpha} = 1-Mh'(\alpha)X. \qquad (2.28)$$

At any wavelet, the ratio of the strain rate to the input strain rate

$$\frac{\partial e}{\partial t}/\frac{\partial e}{\partial t}\bigg|_{X=0} = \frac{\partial e}{\partial X}/\frac{\partial e}{\partial x}\bigg|_{X=0} = p^{-1}.$$

According to linear theory $p \equiv 1$ and the level of the strain rate in the material, like the level of the strain, is bounded by its value at X=0. By contrast the finite rate theory predicts that even though the strain level is still bounded by its level at X=0, because the characteristic wavelets which carry invariant, but different, values of strain may coalesce, the strain rate may become unbounded compared with its level at input. This usually heralds the formation of a shock layer. In this layer the model of the material, defined by the equation of state (2.4), is usually invalid and must be refined to take into account mechanisms which can be safely neglected outside it. Examples of such mechanism are material viscosity and heat conduction. If the transmitting material is an elastic rod or string then the lateral inertia of the material, which can be neglected outside shock layers and away from the driven edge, is also locally important, (see Parker and Varley [2]). If it can be argued, or observed, that the shocks do not produce reflected waves, so that away from shocks the variations in e,u and T are still given by (2.18), (2.19), (2.26) and (2.27), then the shock trajectories and their effect on the deformation can readily be calculated

without, at the same time, determining their structures. The main effect of such weak shocks is to attenuate the amplitude of the disturbance.

3. Weak Shocks

Shocks attenuate a simple wave because the characteristic wavelets, each of which carries a constant value of e and u, coalesce into it. In fact, according to the elastic model, if a shock is allowed to propagate into an infinite region all wavelets except those carrying vanishingly small values of u and e will coalesce into it, or some other shock, so that ultimately the disturbance will be fully attenuated. If $t=S(X)$ denotes the arrival time of a shock at X, if $\alpha^+(X)$ and $\alpha^-(X)$ are the characteristic wavelets immediately ahead and behind the shock, then the condition that a particle position is unchanged by the passage of the shock together with the law relating the change in momentum to the change in traction imply that $u=-e$ both before and after the passage of the shock and that

$$S'(X) = 1-\frac{1}{2}M\{h(\alpha^+)+h(\alpha^-)\}. \qquad (3.1)$$

Also, the condition that the wavelets α^+ and α^- are at the shock at the same time imply that

$$S-X = \alpha^+-Mh(\alpha^+)X = \alpha^--Mh(\alpha^-)X. \qquad (3.2)$$

Once the input signal $h(\alpha)$ is specified, equations (3.1) and (3.2) govern the variations of $S(X)$, $\alpha^+(X)$ and $\alpha^-(X)$ at any shock.

To illustrate the role of shocks as attenuating mechanisms, consider the special case of a shock moving into an undisturbed region so that $h(\alpha^+)\equiv 0$ for $\alpha^+\leq 0$. Then (3.1) and (3.2) integrate to give

$$X = \frac{2}{Mh^2(\alpha^-)}\int_0^{\alpha-} h(s)ds, \quad \text{and} \quad t = X+\alpha^- - Mh(\alpha^-)X \qquad (3.3)$$

as the parametric representation of the shock trajectory. According to (3.3), for a shock to form at the front $\alpha=0$, $Mh'(0)>0$. Then, the point and time of formation are given by

$$X_F = t_F = [Mh'(0)]^{-1} \qquad (3.4)$$

and, by (2.28), at formation $p(o,X_F)=0$ so that the strain rate is unbounded. For definiteness, suppose that $Me(t,0)$ increases monotonically from zero to some maximum value at time t_m and then descreases monotonically to zero at t_o. Then, once formed, the strength of the shock, $Mh(\alpha^-)$, increases from zero until it reaches its maximum strength when it reaches the particle

$$X_m = \frac{2}{Mh^2(\alpha_m)}\int_0^{\alpha_m} h(s)ds. \qquad (3.5)$$

Then, the shock strength decreases until at distances which are large compared with the shock attenuation length

$$\ell_a = 2M\int_0^{t_o} h(s)ds \qquad (3.6)$$

the shock strength

$$Mh(\alpha^-) = (\frac{\ell_a}{X})^{\frac{1}{2}}[1+o(1)] \qquad (3.7)$$

For $X<X_m$, in the pulse generated at $X=0$ over the time interval $0\leq t\leq t_o$

$$\max\{Me(t,X)\} = \max\{Me(t,0)\} \qquad (3.8)$$

so that the amplitude of the pulse is not attenuated. However, for $X>X_m$

$$\max.\{Me(t,X)\} = \text{shock strength} = O[(\frac{\ell_a}{X})^{\frac{1}{2}}], \qquad (3.9)$$

so that ultimately the amplitude of the pulse is vanishingly small. Note that because all the characteristic wavelets except those carrying vanishingly small values of $Mh(\alpha)$ ultimately coalesce into shocks, the deformation outside shock layers is ultimately independent of the details of the input signal $h(\alpha)$: the signal is fully amplitude dispersed. In the far field, between shocks

$$Me \approx \{1 - \frac{t_o - t}{X}\} \qquad (3.10)$$

and, at any X, the profile of Me is an N-wave.

4. Small Non-uniformities

When a disturbance is not adjacent to a uniform region it is no longer generated by a simple wave. However, the deformation at any particle can always be thought of as generated by two progressing waves - an α-wave moving to the right and a β-wave moving to the left. In the α-wave the combination of variables

$$f = \frac{1}{2}[u - c(e)] \qquad (4.1)$$

is invariant at each α-wavelet which propagates so that

$$\left.\frac{DX}{Dt}\right|_{\alpha} = a(e). \qquad (4.2)$$

In the β-wave the combination of variables

$$g = \frac{1}{2}[u + c(e)] \qquad (4.3)$$

is invariant at each β-wavelet which propagates so that

$$\left.\frac{DX}{Dt}\right|_{\alpha} = -a(e). \tag{4.4}$$

If the elastic material occupies the region $0 \le X \le L$ and if

$$\left.\begin{aligned} f &= F(t) \text{ at } X = 0, \\ \text{while} \qquad g &= G(t) \text{ at } X = L, \end{aligned}\right\} \tag{4.5}$$

and if an α-wavelet is tagged by the time it left X=0, while a β-wavelet is tagged by the time it left X=L, then according to (4.1)-(4.5)

$$u = G(\beta)+F(\alpha), \text{ and } c = G(\beta)-F(\alpha). \tag{4.6}$$

If the α-wave is a simple wave then $G(\beta)\equiv 0$ and since a(e) is invariant at an α-wavelet (2.2) integrates to give (2.16) for its arrival time at X.

Classical linear theory takes $a\equiv 1$ in (4.2) and (4.4) to give

$$t = \alpha+X = \beta+L-X \tag{4.7}$$

for the arrival times. The only interaction between the α and β waves is at the boundaries X=0 and X=L. The main mathematical problem of linear theory is to determine the signal functions F(α) and G(β) from prescribed initial and boundary data: this usually involves solving linear difference equations.

If no restriction is placed on the amplitudes of the signal functions F and G there is no general mathematical technique for determining the arrival times. To see how conditions in the α-component of the disturbance are influenced by the β-component, following Mortell and Varley [3] we take (α,X) as the independent variables and c(α,X) together with the incremental arrival time p(α,X) as the

dependent variables. It can be shown that conditions (4.1)-(4.4) imply that

$$\frac{\partial}{\partial X}(\sqrt{a(c)}\ p) = a'(c)a^{-\frac{3}{2}}(c)F'(\alpha) \qquad (4.8)$$

and that

$$\frac{\partial c}{\partial \alpha} + F'(\alpha) = \frac{1}{2}a(c)p\frac{\partial c}{\partial X}, \qquad (4.9)$$

where

$$u = c+2F(\alpha). \qquad (4.10)$$

In (4.8) and (4.9) a is considered a function of c rather than e. The β-component of the disturbance is said not to interact with the α-component if the trajectory of an α-wavelet is determined only by the signal $F(\alpha)$ it carries: then the α-wave is a simple wave. According to the linear approximation there is no such interaction and both the α and β waves are non-dispersed simple waves. This approximation is valid when

(i) $|Me| \ll 1;$

while (4.11)

(ii) both $|MF'(\alpha)| \ll \omega_L$ and $|MG'(\beta)| \ll \omega_L$.

However, according to (4.8), even when the restriction (ii) is dropped, in the small amplitude finite rate limit there is no interaction and

$$t-X = \alpha+F(\alpha)X \qquad (4.12)$$

so the α-wave is still a simple wave even though it is propagating into a non-uniform region. A similar analysis applied to the β-component of the disturbance yields

$$t+(X-L) = \beta+MG(\beta)(X-L) \tag{4.13}$$

in the small amplitude finite rate limit. The description (4.6), (4.12) and (4.13) can be applied to many nonlinear phenomena which occur when the amplitude of the disturbance is small in the sense (4.11) (i). It has been used by Mortell and Varley [4] to discuss resonant oscillations and the decay of standing waves. The determination of the signal functions F and G usually involves solving nonlinear difference equations.

5. Nonplanar Waves

Simple waves need not be unidirectional.† Even though the wavelets, at which the stress, strain and material velocity are invariant, are plane and travel at an invariant velocity, their speed and direction of travel can vary from wavelet to wavelet. Trowbridge and Varley [6] have shown how these simple waves can be modulated to obtain a global statement for conditions in an elastic pulse which is moving into a uniform region. Their analysis is valid in the geometrical acoustics limit when the local frequency of the pulse is large compared with the frequency defined by the local curvature and speed of its front. The technqiues used are an extension to finite amplitude waves of those used in the linear ray theories of geometric optics and acoustics, (see Luneberg [7]).

As an illustration of the theory consider a pulse in a material whose response is elastic and homogeneous with respect to its state before its arrival. Then, if

$$\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X},t)$$

denote the Cartesian co-ordinates (x_i), i=1,2,3, of the particle "$\underset{\sim}{X}$" which prior to the arrival of the pulse had co-ordinates (X_i),

†A general account of such waves in both isotropic and anisotropic elastic materials has been given by Varley [5].

the equations governing the isentropic motion of the material relate the components of the deformation gradient tensor

$$\underset{\sim}{e} = \left(\frac{\partial x_i}{\partial X_j}\right), \quad i,j=1,2,3, \tag{5.2}$$

and the material velocity

$$\underset{\sim}{u} = \left(\frac{\partial x_i}{\partial t}\right), \quad i=1,2,3 \tag{5.3}$$

by the equations

$$\underset{\sim}{C}(\underset{\sim}{e})\underset{\sim}{e},_{\underset{\sim}{X}} = \rho_o\underset{\sim}{u},_t. \tag{5.4}$$

In (5.4)

$$\underset{\sim}{C} = (C_{ijrs}) = \left(\frac{\partial T_{ij}}{\partial e_{rs}}\right), \tag{5.5}$$

where the Piola-Kirchoff stress tensor $\underset{\sim}{T}(\underset{\sim}{e})$ is determined from the stress-strain relations for the material, and ρ_o is the uniform density in the undeformed material. Equations (5.4) are supplemented by the compatibility equations

$$\underset{\sim}{e},_t = \underset{\sim}{u},_{\underset{\sim}{X}} \tag{5.6}$$

which are obtained by eliminating $\underset{\sim}{x}(\underset{\sim}{X},t)$ from (5.2) and (5.3).

A simple wave is composed of a fan of plane wavelets at each of which $\underset{\sim}{e}$ and $\underset{\sim}{u}$ are invariant. A wavelet propagates with an invariant speed which not only depends on the values of $\underset{\sim}{e}$ and $\underset{\sim}{u}$ it carries but also on its direction of propagation. If $\underset{\sim}{n}(\alpha)$ is the unit normal to, and $v(\alpha)$ the speed of travel of, the wavelet α, which is tagged by the time $t=\alpha$ when it passed some reference particle $\underset{\sim}{X}=\underset{\sim}{Y}$, then the trajectory of the wavelet is given by

$$W(t-\alpha) = \underset{\sim}{N}\cdot(\underset{\sim}{X}-\underset{\sim}{Y}) \tag{5.7}$$

where

$$\underset{\sim}{N} = \underset{\sim}{e}^T\underset{\sim}{n}, \quad \text{and} \quad W = v-\underset{\sim}{n}\cdot\underset{\sim}{u}. \tag{5.8}$$

Since $\underset{\sim}{e}(\alpha)$ and $\underset{\sim}{u}(\alpha)$ are invariant at any wavelet α, according to (5.7) in a simple wave

$$\underset{\sim}{e},_{\underset{\sim}{X}}+\underset{\sim}{e},_t\frac{\underset{\sim}{N}}{W} = \frac{D\underset{\sim}{e}}{D\underset{\sim}{X}}\Big|\alpha \equiv 0, \tag{5.9}$$

and

$$\underset{\sim}{u},_{\underset{\sim}{X}}+\underset{\sim}{u},_t\frac{\underset{\sim}{N}}{W} = \frac{D\underset{\sim}{u}}{D\underset{\sim}{X}}\Big|\alpha \equiv 0 \tag{5.10}$$

so that, by (5.6), the variation in $\underset{\sim}{e}$ at a particle is related to the variation in $\underset{\sim}{u}$ by

$$\underset{\sim}{e},_t+\underset{\sim}{u},_t\frac{\underset{\sim}{N}}{W} = 0. \tag{5.11}$$

If (5.9) and (5.11) are used, conditions (5.4) imply that

$$[\underset{\sim}{S}(\underset{\sim}{e},\underset{\sim}{N})-\rho_o W^2\underset{\sim}{I}]\underset{\sim}{u},_t = 0, \tag{5.12}$$

where

$$\begin{aligned} S_{ij}(\underset{\sim}{e},\underset{\sim}{N}) &= C_{irjs}(\underset{\sim}{e})N_rN_s = (C_{irjs}e_{kr}e_{\ell s})n_kn_\ell \\ &= c_{ikj\ell}(\underset{\sim}{e})n_kn_\ell, = s_{ij}(\underset{\sim}{e},\underset{\sim}{n}) \text{ say.} \end{aligned} \tag{5.13}$$

It can be shown that the acoustic tensor $\underset{\sim}{c}$ depends on $\underset{\sim}{e}$ only through its dependence on the Green-strain tensor $\underset{\sim}{G}=\underset{\sim}{e}^T\underset{\sim}{e}$. Its particular form is also dependent on the symmetry properties of the material. According to (5.12), in any simple wave $\rho_o W^2(\underset{\sim}{e},\underset{\sim}{N})$ is an eigenvalue of $\underset{\sim}{S}(\underset{\sim}{e},\underset{\sim}{N})$ and the acceleration $\underset{\sim}{u},_t$ is an associated right eigenvector. In particular, if $\rho_o W^2$ is a simple root of $\underset{\sim}{S}$ then we can write

$$\underset{\sim}{u},_t = c,_t \underset{\sim}{R} \tag{5.14}$$

and, by (5.11) and (5.14),

$$\underset{\sim}{e},_t = -c,_t \frac{\underset{\sim}{R}\underset{\sim}{N}}{W} \tag{5.15}$$

where $\underset{\sim}{R}(\underset{\sim}{e},\underset{\sim}{N})$ is the unit right eigenvector corresponding to the speed $W(\underset{\sim}{e},\underset{\sim}{N})$ and the signal function c is an arbitrary scalar function of α. Once c and $\underset{\sim}{N}$ are specified as functions of $\alpha(=t)$ at $\underset{\sim}{X}=\underset{\sim}{Y}$, equations (5.14) and (5.15) determine the values of u and e carried by each α-wavelet.

The simple wave relations (5.14) and (5.15) also hold at a particle immediately after the passage of an acceleration front, of quite arbitrary shape, which is moving into a uniform region, (see Varley [8]). Such a front is a characteristic surface for the system of quasi-linear hyperbolic equations (5.4) and (5.6). The speed $W(\underset{\sim}{1},\underset{\sim}{N})$ of the front is related to its normal $\underset{\sim}{N}$ by the characteristic condition (5.12) with $\underset{\sim}{e}\equiv\underset{\sim}{1}$. If at t=0 the front coincides with a material surface whose equation is given parametrically by

$$\underset{\sim}{X} = \underset{\sim}{Y}(a,b) \tag{5.16}$$

and if

$$\underset{\sim}{N} = \underset{\sim}{M}(a,b) \tag{5.17}$$

is the normal to this surface at the point (a,b) then, at any subsequent time t, the equation of the front is

$$\underset{\sim}{X} = \underset{\sim}{Y}(a,b)+W,_{\underset{\sim}{N}}(\underset{\sim}{1},\underset{\sim}{M})t. \tag{5.18}$$

According to (5.18), an acceleration front propagates with constant Lagrangian velocity $W,_{\underset{\sim}{N}}(\underset{\sim}{1},\underset{\sim}{M})$ along the bi-characteristic

curve defined by keeping (a,b) constant in (5.18). In the sense that it can be shown (see [8]) that the variation in acceleration at any point moving with velocity $W,_{\underset{\sim}{N}}(\underset{\sim}{1},\underset{\sim}{M})$ can be determined if its value is known at any one time, t=0 say, these rays are the carriers of the disturbance. In analogy with the definition of a photon in the theory of optics, the "particle" (a,b), whose trajectory at the front where $\underset{\sim}{e}=\underset{\sim}{1}$ is given by (5.18), may be called an <u>elaston</u>. The velocity of an elaston is always normal to the front for an isotropic material, it is not generally so for an anisotropic material.

The simple wave relations (5.14) and (5.15) also hold to a first approximation in any high frequency pulse behind the front (5.18). There the signal is carried by the one parameter family of characteristic wavelets which at some previous time coincided with the material surface (5.16). The normal speed, $w(\underset{\sim}{e},\underset{\sim}{n})$, relative to the material of any such wavelet is determined by the local deformation $\underset{\sim}{e}$ and by its normal $\underset{\sim}{n}$ from the characteristic condition

$$\det|\underset{\sim}{s}(\underset{\sim}{e},\underset{\sim}{n})-\rho_0 w^2\underset{\sim}{1}| = 0. \qquad (5.19)$$

Actually, conditions at any point (a,b) on the material surface (5.16) at time t=α only influence the deformation at points on the trajectory

$$\underset{\sim}{x} = \underset{\sim}{x}(a,b,\alpha:t) \qquad (5.20)$$

of the elaston (a,b,α), which is labelled by the time t=α and the point (a,b) at which it crossed the material surface (5.16). It can be shown (see Varley and Cumberbatch [9]) that the trajectory of the elaston is determined from the bi-characteristic relations

$$\frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{u}+\underset{\sim}{w},_{\underset{\sim}{n}}(\underset{\sim}{e},\underset{\sim}{n}) \qquad (5.21)$$

where the variation in $\underset{\sim}{n}(a,b,\alpha:t)$ is given by

$$\frac{d\underset{\sim}{n}}{dt} = (\underset{\sim}{n}\underset{\sim}{n}-\underset{\sim}{1})W,_{\underset{\sim}{e}}(\underset{\sim}{e},\underset{\sim}{n})\underset{\sim}{e},_{\underset{\sim}{x}}. \tag{5.22}$$

In (5.21) and (5.22) $\underset{\sim}{u}$ and $\underset{\sim}{e}$ are considered as functions of $(\underset{\sim}{x},t)$. Since w is homogeneous of degree one in $\underset{\sim}{n}$,

$$\underset{\sim}{n}\cdot w,_{\underset{\sim}{n}} = w \tag{5.23}$$

so, by (5.21),

$$\underset{\sim}{n}\cdot\frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{u}\cdot\underset{\sim}{n}+w \tag{5.24}$$

and the elaston stays on the characteristic wavelet α=constant. The trajectory of the elaston in particle space

$$\underset{\sim}{X} = \underset{\sim}{X}(a,b,\alpha:t) \tag{5.25}$$

is conveniently described in terms of

$$w = W(\underset{\sim}{e},\underset{\sim}{N}), \tag{5.26}$$

where

$$\underset{\sim}{N} = \underset{\sim}{e}^T\underset{\sim}{n}. \tag{5.27}$$

At an elaston

$$\frac{d\underset{\sim}{X}}{dt} = W,_{\underset{\sim}{N}}(\underset{\sim}{e},\underset{\sim}{N}), \tag{5.28}$$

where the variation in

$$\hat{N}(a,b,\alpha:t) = \underset{\sim}{N}/|\underset{\sim}{N}| \tag{5.29}$$

is given by

$$\frac{d\hat{\underset{\sim}{N}}}{dt} = (\hat{\underset{\sim}{N}}\hat{\underset{\sim}{N}}-\underset{\sim}{1})W,_{\underset{\sim}{e}}(\underset{\sim}{e},\underset{\sim}{N})\underset{\sim}{e},_{\underset{\sim}{X}} \tag{5.30}$$

where now $\underset{\sim}{e}$ is considered as a function of $(\underset{\sim}{X},t)$.

In the finite amplitude theory the trajectory of an elaston can only be determined from (5.21) and (5.22) if at the same time the deformation $\underset{\sim}{u}(\underset{\sim}{x},t)$ and $\underset{\sim}{e}(\underset{\sim}{x},t)$ are also determined. The linear theory however neglects the influence of the deformation on the trajectory of the elaston and formally takes

$$\underset{\sim}{e} \equiv \underset{\sim}{1}, \text{ and } \underset{\sim}{u} \equiv 0 \tag{5.31}$$

in (5.21) and (5.22) which then integrate to give

$$\underset{\sim}{n} = \underset{\sim}{M}(a,b) \tag{5.32}$$

and

$$\underset{\sim}{x} = \underset{\sim}{Y}(a,b)+w,_{\underset{\sim}{n}}(1,\underset{\sim}{M})(t-\alpha) \tag{5.33}$$

as the trajectory of an elaston. In addition, according to linear theory (5.14) and (5.15) integrate to give

$$\underset{\sim}{u} = -c\underset{\sim}{R}(\underset{\sim}{1},\underset{\sim}{M}), \tag{5.34}$$

and

$$\underset{\sim}{e}-\underset{\sim}{1} = c\underset{\sim}{R}(\underset{\sim}{1},\underset{\sim}{M})\underset{\sim}{M}/W(1,\underset{\sim}{M}). \tag{5.35}$$

The statements (5.33)-(5.34) provide a complete parametric description of conditions in a high frequency pulse once the signal function $c(a,b,\alpha:t)$ is known. It can be shown that according to linear theory the variation in c at an elaston satisfies an equation of the form

$$\frac{dc}{dt} + A(a,b,t)c = 0 \tag{5.36}$$

where the function A is determined by the material properties, and by the local geometry of the surface (5.16) at (a,b). Once the variation in c at the surface (5.16),

$$c = f(a,b,t), \tag{5.37}$$

is specified, equation (5.36) is integrated subject to the initial condition that at $t=\alpha$

$$c = f(a,b,\alpha), \tag{5.38}$$

As a simple illustration which exhibits some of the important differences between the predictions of the linear and the finite rate theories, we consider a pulse which is generated in an isotropic material when the boundary surface (5.16) is suddenly loaded by a time varying normal traction which may also vary with (a,b). Such a situation occurs when the surface is impacted by a pressure wave. For an isotropic material the local speed w is always of the form

$$w = U(e_1,e_2,e_3)(\underset{\sim}{n}\cdot\underset{\sim}{n})^{\frac{1}{2}} \tag{5.39}$$

where U is a symmetric function of (e_1,e_2,e_3)—the eigenvalues of the strain tensor $(\underset{\sim}{e}\underset{\sim}{e}^T)^{\frac{1}{2}}-1$. In a pulse generated by a varying normal pressure the normal to the characteristic wavelets $\underset{\sim}{n}$ is a principal direction. The rate of stretching in this direction is also much more rapid than in any other direction so that, to a first approximation,

$$e_2 = e_3 = 1: \tag{5.40}$$

$$\underset{\sim}{R} = \underset{\sim}{n} = \underset{\sim}{n}(a,b), \text{ and } \underset{\sim}{N} = (1+e)\underset{\sim}{n} \tag{5.41}$$

so that both the particle and elaston paths are along fixed rays which are normal to the impacted surface (5.16). Equations (5.14) and (5.15) integrate to give

$$\tilde{e} = \tilde{1} + e\tilde{n}\tilde{n} \tag{5.42}$$

and

$$\tilde{u} = -c(e)\tilde{n} \tag{5.43}$$

where c(e) is given by (2.13) with T(e) interpreted as the principal stress in the $\tilde{n}$-direction when $e_2 = e_3 = 1$. In terms of T(e) the speed of the front U is given by

$$\rho_0 U^2 = (1+e)^2 T'(e): \tag{5.44}$$

the trajectory of an elaston is given by

$$\frac{d\tilde{x}}{dt} = [U(e) - c(e)]\tilde{n}(a,b), \tag{5.45}$$

or, in particle space, by

$$\frac{D\tilde{X}}{Dt} = a(e)\tilde{n}(a,b). \tag{5.46}$$

The strain field is given by

$$\tilde{g} = \tilde{e}\tilde{e}^T = \tilde{1} + [(1+e)^2 - 1]\tilde{n}\tilde{n} \tag{5.47}$$

and the stress field by

$$\tilde{t} = (T-S)\tilde{1} + S\tilde{n}\tilde{n} \tag{5.48}$$

In (5.48) the material function T(e) has already been defined and the material function $\frac{1}{2}S(e)$ is the maximum shear stress at any point. Note that both T and S can be determined from the behaviour of the material in the uniaxial deformations discussed in section 3.

It remains to determine the variation in e and to relate it to its specified variation

$$e = H(a,b,t) \text{ on } \underset{\sim}{X} = \underset{\sim}{Y}(a,b). \tag{5.49}$$

Once e is determined equations (5.45) and (5.46) can be integrated to obtain $\underset{\sim}{x}$ and $\underset{\sim}{X}$ as functions of (a,b,α,t). Trowbridge and Varley [6] show that the variation in e at an elaston is 6
governed by a nonlinear first order ordinary differential equation - the <u>nonlinear transport equation</u>. Moreover, in terms of the distance measure X, which varies at an elaston so that

$$\frac{dX}{dt} = a(e) \tag{5.50}$$

and satisfies the auxiliary condition that

$$X = 0 \text{ on } \underset{\sim}{X} = \underset{\sim}{Y}(a,b), \tag{5.51}$$

this equation integrates to give a relation of the form

$$G(e) = H(a,b,\alpha)\left(1+\frac{X}{\rho_1}\right)^{-\frac{1}{2}}\left(1+\frac{X}{\rho_2}\right)^{-\frac{1}{2}} \tag{5.52}$$

for the variation in e. In (5.52), G(e) is a material function $\rho_1(a,b)$ and $\rho_2(a,b)$ are the principal radii of curvature at the particle (a,b) of the material surface $\underset{\sim}{X} = \underset{\sim}{Y}(a,b)$ before it is loaded, and $H(a,b,\alpha)$ is determined from the variation of e at this surface as

$$H = G(h(a,b,\alpha)). \tag{5.53}$$

In terms of X, equations (5.46) integrate to give

$$\underset{\sim}{X} = X\underset{\sim}{n}(a,b)+\underset{\sim}{Y}(a,b). \tag{5.54}$$

Once $e(a,b,\alpha:X)$ is determined from (5.52), the equation

$$\frac{dt}{dx} = [a(e)]^{-1} \tag{5.55}$$

can be integrated, subject to the condition that

$$t = \alpha \text{ when } X = 0, \tag{5.56}$$

to give $t(a,b,\alpha:X)$ which, together with equations (5.42), (5.43), (5.47), (5.48), (5.52) and (5.54), give a complete (parametric) description of the velocity, strain, and stress field in the high frequency pulse.

In the small amplitude finite rate limit, (5.52) implies that

$$e = h(a,b,\alpha)\left(1+\frac{X}{\rho_1}\right)^{-\frac{1}{2}}\left(1+\frac{X}{\rho_2}\right)^{-\frac{1}{2}} \tag{5.57}$$

while (5.43) and (5.48) are approximated by

$$\underset{\sim}{u} = -e\underset{\sim}{n}, \text{ and } \underset{\sim}{t} = (\lambda\underset{\sim}{1}+2\mu\underset{\sim}{n}\underset{\sim}{n})e. \tag{5.58}$$

where λ and μ are the Lamé constants. Equations (6.54) are unchanged and (5.54) implies that

$$t-X = \alpha - Mh(a,b,\alpha)\int_0^X \frac{ds}{\sqrt{(1+\frac{s}{\rho_1})(1+\frac{s}{\rho_2})}}, \tag{5.59}$$

where t is normalized as in (2.7) with $E=\lambda+2\mu$. The classical linear theory would take

$$\alpha = \alpha_L = t-X \tag{5.60}$$

in (5.59)

As an illustration of the differences between the predictions of the linear and finite rate theories consider the case when the loaded boundary is a sphere of unloaded radius R_o. Then

$$\rho_1 = \rho_2 = R_o \tag{5.61}$$

and (5.57) reads

$$e = h(a,b,\alpha)(\frac{R}{R_o})^{-1} \quad (5.62)$$

where

$$R = R_o+X \quad (5.63)$$

is the radial distance to the particle $\underset{\sim}{X}$ before the arrival of the pulse. Equation (5.59) reads

$$t-(R-R_o) = \alpha-Mh(a,b,\alpha)R_o \ln(\frac{R}{R_o}), \quad (5.64)$$

$$= \alpha-MeR\ln(\frac{R}{R_o}). \quad (5.65)$$

According to both theories the radial spread of the pulse attenuates the disturbance. However, whereas the linear theory predicts that the amplitude of the strain rate will always decrease in the pulse, the nonlinear theory predicts that in that part of the pulse where $M\frac{\partial h}{\partial \alpha}>0$ the strain rate will increase and ultimately shocks will form. In particular, a shock will form at the front $\alpha=0$ at a radial distance along the ray (a,b) given by

$$\ln\frac{R}{R_o} = R_oM\frac{\partial h}{\partial \alpha}(a,b,0). \quad (5.66)$$

Another important difference between the predictions of the two theories is that whereas the linear theory predicts that the decay in e is like $(\frac{R}{R_o})^{-1}$ with a coefficient which depends on the detailed loading pattern, the nonlinear theory predicts that as $R/R_o\to\infty$, away from shocks, the pulse is fully amplitude dispersed and

$$Me = (1-\frac{t-t_o}{R})(\ln\frac{R}{R_o})^{-1}[1+0(1)] \quad (5.67)$$

so that the asymptotic variation of e is independent of the

detailed loading pattern.

For a more detailed account of the theory and its applications the reader is referred to [6].

6. Effect of Locally Small Damping.

The nonlinear elastic model, which neglects all attenuating mechanisms outside shock layers, predicts that the level of the strain rate will always increase in distance from the driven edge, and that shocks will always form, in any unidirectional deformation generated by a cyclic loading of the boundary. This is because in any such loading $Mh'(\alpha)>0$ at some of the wavelets which leave the boundary in each cycle. Actually, in practice, the strain rate will only increase if the amplitude of the applied strain rate is above some critical value. Varley and Rogers [10] and Seymour and Varley [11] have discussed in great detail the effect of locally small damping in the small amplitude finite rate limit in situations when it might be thought that the response of the material is elastic. Again the deformation is considered to be generated by a slowly modulated simple wave. We briefly review their results for periodic deformations.

To model the damping mechanisms, the equation of state (2.4) is replaced by an equation of state which relates the stress and strain rates at a particle to the current values of stress and strain by

$$\frac{\partial T}{\partial t} = \phi(T,e)\frac{\partial e}{\partial t} + \psi(T,e), \tag{6.1}$$

where ϕ and ψ are material functions. We consider high frequency deformations for which

$$\left|\frac{\partial e}{\partial t}\right| >> \left|\frac{\psi(T,e)}{\phi(T,e)}\right| \tag{6.2}$$

so that, to a first approximation, the stress variation can

be calculated from the strain variation at a particle from the elastic law

$$\frac{dT}{de} = \phi(T,e), \tag{6.3}$$

and the effect of the damping term ψ is locally small. In any time periodic deformation the mean of T, T_m, is independent of X. In any <u>small amplitude</u> periodic deformation the mean of e, e_m, is also independent of X and is related to T_m by the static law

$$\psi(T_m,e_m) = 0. \tag{6.4}$$

If, for convenience, we measure T and e so that

$$T_m = e_m = 0, \tag{6.5}$$

Seymour and Varley [11] show that to a first approximation in the limit (6.2), when the effect of dissipation is locally small, T can be calculated from e by the linear elastic law

$$T = \phi(0,0)e \tag{6.6}$$

and that u can be calculated from e by the relation

$$u = -a_m e \tag{6.7}$$

where

$$a_m = \sqrt{\frac{\phi(0,0)}{\rho_0}} . \tag{6.8}$$

is the mean sound speed. Relations (6.6) and (6.7) also hold in an elastic wave. However, the variation in e(X,t) predicted by the model (2.1) is not, in general, given by (2.26) and (2.27)

In terms of the mean attenuation length

$$\ell_m = -2a_m\left(\frac{\phi}{\phi\psi,_\sigma+\psi,_\lambda}\right), >0, \qquad (6.9)$$

where ϕ and the derivatives of ψ are evaluated at (0,0),

$$e = h(\alpha)e^{-X/\ell_m} \qquad (6.10)$$

where

$$t-X/a_m = \alpha-\left(\frac{M\ell_m}{a_m}\right)h(\alpha)(1-e^{-X/\ell_m}). \qquad (6.11)$$

In (6.11) the parameter

$$M = (2\phi)^{-1}(\phi\phi,_\sigma+\phi,_\lambda): \qquad (6.12)$$

according to the dynamic elastic law (6.3)

$$T = \phi(0,0)e[1+Me+0(e^2)]. \qquad (6.13)$$

Conditions (6.10) and (6.11) predict that the level of the strain rate can only begin increasing with X at X=0 if at X=0

$$\frac{\partial e}{\partial t} > \frac{a_m}{M\ell_m}, = \omega_c \text{ say}. \qquad (6.14)$$

If

$$\left|\frac{\partial e}{\partial t}\right| << |\omega_c|, \qquad (6.15)$$

then the linear dissipative theory which gives

$$e = h(t-X/a_m)e^{-X/\ell_m} \qquad (6.16)$$

is uniformly good approximation to the finite rate theory with dissipation. If, in addition,

$$X << \ell_m \tag{6.17}$$

the predictions of the linear elastic model, which is

$$e = h(t - X/a_m), \tag{6.18}$$

is also good. The nonlinear elastic model is good when

$$X << \ell_m \tag{6.19}$$

for imposed strain rates

$$\left|\frac{\partial e}{\partial t}\right| >> |\omega_c|. \tag{6.20}$$

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13. ABSTRACT

This paper describes some of the techniques which are currently being used to investigate finite amplitude waves in elastic and viscoelastic materials. In particular we show how the simple wave solutions, which describe finite amplitude plane progressing waves in elastic materials, may be modified to describe the effects of reflection from boundaries, deformations behind curved wave fronts, and the effects of locally small damping mechanisms.

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